1. (10 points) If $A \in \mathcal{R}^{n\times n}$ is symmetric and positive definite, show that

$$|a_{i,j}| \leq \frac{1}{2}\left(a_{i,i} + a_{j,j}\right)$$

holds for all $1 \leq i, j \leq n$.

2. (10 points) Let $A \in C^{m \times n}$ with $m \geq n$. Show that A^*A is nonsingular if and only if A has full rank.

3. (10 points) Let $A \in C^{m\times m}$, and let a_j be its jth column. Prove the following inequality:

$$|\det(A)| \leq \prod_{j=1}^{m} \|a_j\|_2.$$

4. Let

$$T = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}.$$

(a) (5 points) Use Gerschgorin's disk theorem to show that T is positive semidefinite.

(b) (5 points) Without computing the eigenvalues, explain why these eigenvalues are real.

5. Let

$$B = \begin{pmatrix} 3 & -6 & 41/5 \\ 0 & 1 & 1 \\ 4 & -8 & 63/5 \end{pmatrix}.$$

   (a) (10 points) Find the QR decomposition of B using the Gram-Schmidt orthogonalization process.

   (b) (10 points) Write down the unitary Householder reflector matrix that you should multiply against B in order to zero out all but the first entry of its first column.

6. Consider the following linear system,

$$A\mathbf{x} = F, \tag{1}$$

where

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 0 & -1 & 2 & -1 \\ \cdots & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}$$

(a) (10 points) Prove that the $n \times n$ tridiagonal matrix A is symmetric, positive definite (SPD).

(b) (10 points) Let B be a tridiagonal SPD matrix in the form of the matrix A. Prove that the Cholesky factor L of B has nonzero entries only along the main diagonal and the sub-diagonal lines, where $B = LL^t$. Give the formula for L.

(c) (10 points) Design an $O(n)$ algorithm to solve the linear system $A\mathbf{x} = F$.

7. (10 points) Let $\mathbf{x}_j \in \mathcal{R}^m$ be the j-th column of $X \in \mathcal{R}^{m\times n}$ be given. Let $\mathbf{y} \in \mathcal{R}^m$ and $\lambda > 0$ also be given. Given a vector $\mathbf{w} \in \mathcal{R}^n$, define the following function

$$J(\mathbf{w}) = \|X\mathbf{w} - \mathbf{y}\|_2^2 + \lambda\|\mathbf{w}\|_1,$$

where $\|\cdot\|_2$ and $\|\cdot\|_1$ denote the 2- and 1-norm in $\mathcal{R}^n$, respectively. Letting the i-th component w_i of $\mathbf{w}$ vary and the other components of $\mathbf{w}$ be fixed, consider the following one-variable minimization problem reduced from $J(\mathbf{w})$:

$$\begin{aligned}\min_{w_i} f(w_i) &\equiv \min_{w_i} \|\sum_{j=1}^{n} w_j\mathbf{x}_j - \mathbf{y}\|_2^2 + \lambda|w_i| + \lambda\sum_{j\neq i}|w_j| \\ &= \min_{w_i} \|w_i\mathbf{x}_i + \mathbf{r}\|_2^2 + \lambda|w_i| + C \\ &= \min_{w_i} \sum_{j=1}^{m}(w_i x_{ji} + r_j)^2 + \lambda|w_i| + C, \end{aligned} \tag{2}$$

where $\mathbf{r} \equiv \sum_{j\neq i} w_j\mathbf{x}_j - \mathbf{y}$ is in $\mathcal{R}^m$ with $\mathbf{r} = (r_k)_{m\times 1}$, and $C = \lambda\sum_{j\neq i}|w_j|$. Show that the optimal solution w_i^* for the minimization problem (2) is given by

$$w_i^* = \begin{cases} 0 & \text{if} \quad |a| \le \lambda, \\ \frac{-\lambda+a}{b} & \text{if} \quad \frac{-\lambda+a}{b} > 0, \\ \frac{\lambda+a}{b} & \text{if} \quad \frac{\lambda+a}{b} < 0, \end{cases}$$

where $a = -\sum_{j=1}^{m} 2x_{ji}r_j$ and $b = \sum_{j=1}^{m} 2x_{ji}^2$.